MATHEMATICS Questions

Answer ANY THREE questions.

1 **(a)** Given $\boldsymbol{A} = \begin{bmatrix} 3 & 3 \\ 1 & 8 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} -3 & 3 \\ 1 & 4 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} 16 & 4 \\ 4 & 1 \end{bmatrix}$ find, where feasible,

(i) $\boldsymbol{A} - \boldsymbol{B}$ (ii) $\boldsymbol{B}^T$

(iii) $\boldsymbol{AB}$ (iv) $\boldsymbol{AC}$

(v) $|\boldsymbol{A}|$ (vi) $|\boldsymbol{C}|$

(vii) $\boldsymbol{A}^{-1}$ (viii) $\boldsymbol{D}^{-1}$

(ix) Show that $\boldsymbol{AA}^{-1} = \boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A}$

(b) Given $\boldsymbol{E} = \begin{bmatrix} 5 & -2 & 0 \\ 3 & 1 & 2 \\ 5 & 3 & 5 \end{bmatrix}$, what is the rank of $\boldsymbol{E}$?

Hint: you can use any suitable method.

(c) Assess whether the following expressions are quadratic form polynomials. Find, for the expressions that **are**, whether they are a positive or negative definite.

(i) $q_1 = -3x^2 - y^2 + 6xz - 2z^2$

(ii) $q_2 = 3x^2 + 10yx + 3xy^2$

(iii) $q_3 = 4x^2 + 2y^2 + 8xy$

2. **(a)** Solve using matrix inversion only **or otherwise**, the following system of equations

(i)
$$\begin{aligned} 10x_1 - 4x_2 + 6x_3 &= 12 \\ 2x_1 + 3x_2 - 5x_3 &= 0 \\ 4x_1 - 5x_2 + 6x_3 &= 5 \end{aligned}$$

(ii) IS: $0.3Y = 252 - 100r$
LM: $0.25Y = 176 + 200r$

(b) Solve the system of equations in (a) (ii) by Cramer's rule.

3 **(a)** For each of the following, find $\frac{dy}{dx}$

(i) $y = 2x^6 - 3x + \frac{1}{x^4} + 42$

(ii) $y = 4e^{x^2+7x}$

(iii) $y = \frac{4e^x - x^5}{x^3+2}$

(iv) $y = (x-3)^3 \ln(2x)$

(v) $y = \ln(6x^2 - 5x)$

(b) Find the stationary value(s) of the following functions, checking for a relative (local) maximum, minimum or point of inflection.

(i) $y = -(x - 10x)^5$

(ii) $y = -2x^2 + 3x + 10$

(c) A firm has total revenue and total cost functions given by:

$R = 200Q - 3Q^2$

and $C = 3Q^2 + 20Q + 40$.

(i) Write the firm's profit function;

(ii) Find the level of output at which profit is maximized and the maximum profit.

4. **(a)** For the following function, find all the first-and second-order partial derivatives:

$q = x^5 + x^2y^5 + 2y^6z + z^2$

(b) Given the function:

$z = x^2 - xy + 2y^2 - 6y + 47$

(i) Find the critical values associated with any optimum;

(ii) By evaluating the Hessian determinant, interpret the second order condition.

(Continued on next page)

4. continued

(c) A monopolist produces two goods x, y with prices p_x, p_y respectively. The demand functions for each good are

$$x = 40 - 0.5p_x \quad \text{and} \quad y = 32 - p_y$$

and the total cost function is

$$TC = 2x^2 + 2xy + y^2 + 20 \quad .$$

(i) Show that the monopolist's profit function is $\prod = 80x - 4x^2 + 32y - 2y^2 - 2xy - 20$.
(ii) Find the level of x and y associated with any optimum;
(iii) Interpret the second-order sufficient condition;
(iv) What prices should be charged for the two goods in orderto maximise profits?

5 **(a)** Given the objective function $z = 5x^2 - 3xy + 4y^2$ subject to the linear constraint $x + y = 10$

(i) Find the pair of x and y values for which z is stationary;
(ii) Interpret the second order conditions of this constrained optimization problem.

(b) A consumer's utility function is given by:

$$U = (x + 2)(y - 2)$$

The price of goods x and y are £8 and £2 per unit respectivelyand the consumers income is £100.

(i) Find the quantities of the two goods which maximize the consumer's utility subject to his/her income, using the Lagrangian method.
(ii) What is the total utility attained by the consumer?
(iii) Evaluate and interpret the determinant of the bordered Hessian determinant.

Statistics Questions

Answer ANY FOUR questions.

1. Answer **ALL** parts of this question.
    a. Consider the following sample of students' marks for a Development Economics module:

Student	1.	2.	3.	4.	5.	6.	7.	8.
Dev Econ mark	63	68	72	78	70	72	74	56

        i. Compute the mean, median, and mode of the sample.
        ii. Compute the range, variance, standard deviation, and coefficient of variation of the sample.

    b. Analyze briefly the relative strengths of each of the three measures of central tendency.

    c. Consider the marks for a Macroeconomics paper for the **same** eight students as above:

Student	1.	2.	3.	4.	5.	6.	7.	8.
Macro mark	57	73	72	76	65	63	64	60

    The covariance between Development Economics marks and Macroeconomics marks is calculated to be 36.64. The correlation coefficient is 0.65 . Briefly interpret these two measures of association and explain their differences. What do these measures say about the students' performance in the two courses?

2. Answer **THREE** of the following questions.
    a. State the relationships between the joint, marginal and conditional probabilities of two events, A and B. Use these to derive and state Bayes' Theorem.

    b. Explain how the F distribution relates to the Chi-squared distribution, and give an example of when you might have to use the t distribution.

    c. Explain
        i. the difference between *standard deviation* and *standard error*.
        ii. the Central Limit Theorem (you do not need to prove it).

    d. Define and briefly explain the following concepts
        i. Sampling error,
        ii. Bias of an estimator,
        iii. Mean Squared Error.

    e. Define Type I and Type II error. Can we reduce the probability of one without increasing the probability of the other?

3. Answer **ALL** parts of this question.

   Let X and Y be normally distributed random variables, with
   $E(\mathrm{X}) = \mu_{XX} \quad E(Y) = \mu_{\mathrm{F}} \quad Var(\mathrm{X}) = \sigma^2_{XX} \quad Var(Y) = \sigma^2_{\mathrm{F}} \quad Cov(\mathrm{X}, Y) = \sigma_{XX\mathrm{F}} = 0$

   a. A random sample of size n is drawn from Y with sample mean
      $\mu_{\mathrm{F}} = \frac{\sum_{ii=1}^{n} y_{ii}}{\boldsymbol{n}}$.

      Show that $Var(\hat{\mu}_{\mathrm{F}}) = \sigma^2/n$.

   b. For constant a, b, show that:

$Cov(a + b\mathrm{X}, Y) = b\ Cov(\mathrm{X}, Y)$.

   c. For constants a, b show that

$Var(a\mathrm{X} + bY) = a^2\sigma^2_{XX} + b^2\sigma^2_{\mathrm{F}} + 2ab\sigma_{XX\mathrm{F}}$

   d. Can we infer that X and Y are independent? Motivate your answer.

4. Answer **ALL** parts of this question.

   a. A random sample of 450 students were asked how long it takes them to go to the university campus. The sample mean travelling time was 35 minutes ($\hat{\mu} = 35$), with a sample standard deviation of $s = 12$ minutes.

      i. Construct a 95% confidence interval for the average travelling time and explain what it shows.
      ii. Assuming that travel time is normally distributed, what is the probability that a randomly chosen student has a travelling time of one hour or greater?

   b. When this research was conducted ten years ago the average travelling time was 40 minutes. Now the Student Union is investigating whether students' travel time has declined since then.

      i. How would you test this hypothesis using the random sample above? What would the appropriate null and alternative hypotheses be?
      ii. Test the hypothesis at the 1% significance level and interpret the results.

5. Answer **ALL** parts of this question.

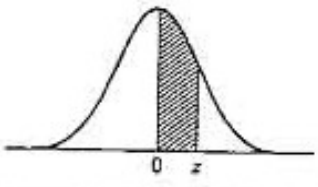
   In a survey of television watching habits, it emerged that 65% of adults in the UK watched at least one football match on TV in the past year. Furthermore, 24% of adults watched at least one cricket match, while 15.6% of adults watched at least one football match and one cricket match.

   a. What is the probability that a person selected at random has not watched a football match on TV in the past year?
   b. What is the probability that a person has not watched a football match, but has watched a cricket match on TV in the past year?
   c. What is the probability that someone has watched at least one match of one sport, and only one of the two sports?
   d. Given that someone has watched a cricket match, what is the probability that he/she has also watched a football match on TV in the past year?
   e. Are the two sports' watching habits independent? Prove your answer.

Table A-1

AREAS UNDER THE NORMAL CURVE

An entry in the table is the proportion under the entire curve which is between $z = 0$ and a positive value of z. Areas for negative values of z are obtained by symmetry.



Second decimal place of z

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

From Paul G. Hoel, *Elementary Statistics*, 3rd ed., © 1971, John Wiley and Sons, Inc., New York, p. 287.

Table A-2

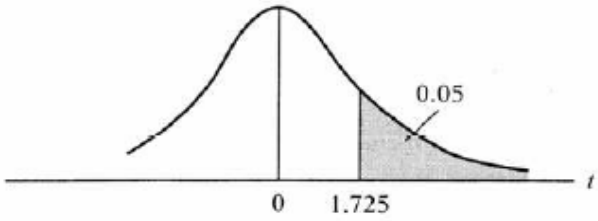
Percentage points of the t distribution

Example

$\Pr(t > 2.086) = 0.025$

$\Pr(t > 1.725) = 0.05$ for df = 20

$\Pr(|t| > 1.725) = 0.10$



df \ Pr	0.25 0.50	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.001 0.002
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	0.816	1.886	2.920	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.214
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
16	0.690	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	1.330	1.734	2.101	2.552	2.878	3.610
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.080	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.500	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.060	2.485	2.787	3.450
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.310	1.697	2.042	2.457	2.750	3.385
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
60	0.679	1.296	1.671	2.000	2.390	2.660	3.232
120	0.677	1.289	1.658	1.980	2.358	2.617	3.160
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.090

Note: The smaller probability shown at the head of each column is the area in one tail; the larger probability is the area in both tails.

Source: From E. S. Pearson and H. O. Hartley, eds., *Biometrika Tables for Statisticians*, vol. 1, 3d ed., table 12, Cambridge University Press, New York, 1966. Reproduced by permission of the editors and trustees of *Biometrika*.